Quantum Field Theory

Set 5: solutions

Exercise 1

Given the group properties we know that $g(p(\alpha,\beta)) \in \mathcal{G}$, so it is a function

$$p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n.$$

Given the assumption that $\mathcal{G}$ is a Lie group we know that this function is smooth and thus it can be expanded in series in a finite neighborhood of the origin. Let's start by using the property of existence of the identity element e. In formulas we have

$$g(\alpha) = g(\alpha) \cdot e = g(\alpha) \cdot g(0) = g(p(\alpha, 0))$$

and

$$g(\alpha) = e \cdot g(\alpha) = g(0) \cdot g(\alpha) = g(p(0, \alpha))$$

This means that when one of the two argument is zero we have

$$p^i(\alpha, 0) = p^i(0, \alpha) = \alpha^i$$

where we have introduced the index i of $\mathbb{R}^n$. The most general series expansion for p^i around the origin is then

$$p^i(\alpha,\beta) = \alpha^i + \beta^i + T^i_{ab}\alpha^a\beta^b + B^i_{abc}\alpha^a\alpha^b\beta^c + D^i_{abc}\alpha^a\beta^b\beta^c + O((\alpha,\beta)^4)$$

where B^i_{abc} is symmetric in the indices a and b while D^i_{abc} is symmetric in b and c. Note that terms like α^2 are excluded by the condition we found earlier from the property of the identity element. For what follows we will only need the expansion up to the second order, so we will neglect the tensors B and D.

The tensor T^i_{ab} has in principle no symmetry properties in the two indices a and b, but we will see now that the symmetric part can always be eliminated by a change of coordinates, while the antisymmetric part can't. Note that since the choice of coordinates in a manifold is arbitrary, this means that the symmetric part of T cannot contain any information about the group structure. In this new coordinate system the product will then look like

$$p'^i(\alpha',\beta') = \alpha'^i + \beta'^i + T^i_{[ab]}\alpha'^a\beta'^b + O((\alpha,\beta)^3,$$

where we have defined the antisymmetric part of T^i_{ab} as

$$T^i_{[ab]} \equiv \frac{T^i_{ab} - T^i_{ba}}{2} = -T^i_{[ba]}.$$

A generic change of coordinates expanded up to the quadratic order in the expansion has the form

$$\alpha'^i = \alpha^i + \delta^i_{ab}\alpha^a\alpha^b + O(\alpha^3).$$

Note that δ^i_{ab} is symmetric in a and b by definition. It's then natural to try to find a particular form for δ for which the symmetric part of T cancels out. Since α, β and p are all coordinates in the manifold, they will all transform in the same way. Namely

$$\beta'^i = \beta^i + \delta^i_{ab}\beta^a\beta^b + O(\beta^3).$$

and

$$p'^i(\alpha',\beta') = p^i(\alpha,\beta) + \delta^i_{ab}p^a(\alpha,\beta)p^b(\alpha,\beta) + O(p(\alpha,\beta)^3).$$

Let's now expand the right hand side of this equation and express it in terms of α' and β'. First of all we need to invert the change of variable to express α as a function of α'. We can do this iteratively since we are working at the quadratic order in the expansion

$$\alpha^i = \alpha'^i - \delta^i_{ab}\alpha^a(\alpha')\alpha^b(\alpha') + O(\alpha'^3) = \alpha'^i - \delta^i_{ab}\alpha'^a\alpha'^b + O(\alpha'^3),$$

where in the last equality we have substituted α using the first equality. Using this result, the product function in the new coordinates then looks like

$$
\begin{aligned}
p'^i(\alpha', \beta') =& p^i(\alpha, \beta) + \delta^i_{ab} p^a(\alpha, \beta) p^b(\alpha, \beta) + O(p(\alpha, \beta)^3) \\
=& \alpha^i + \beta^i + T^i_{ab}\alpha^a\beta^b + \delta^i_{ab}(\alpha^a + \beta^a)(\alpha^b + \beta^b) + O((\alpha, \beta)^3) \\
=& (\alpha'^i - \delta^i_{ab}\alpha'^a\alpha'^b) + (\beta'^i - \delta^i_{ab}\beta'^a\beta'^b) + T^i_{ab}\alpha'^a\beta'^b + \delta^i_{ab}(\alpha^a + \beta'^a)(\alpha'^b + \beta'^b) + O((\alpha', \beta')^3) \\
=& \alpha'^i + \beta'^i + T^i_{ab}\alpha'^a\beta'^b + \delta^i_{ab}(\alpha^a\beta'^b + \alpha'^b\beta'^a) + O((\alpha', \beta')^3)
\end{aligned}
$$

Decomposing T into symmetric and antisymmetric pieces,

$$
T^i_{ab} = T^i_{(ab)} + T^i_{[ab]},
$$

where we have defined the symmetric part of T as

$$
T^i_{(ab)} \equiv \frac{T^i_{ab} + T^i_{ba}}{2} = T^i_{(ba)},
$$

we can write p' as

$$
p'^i(\alpha', \beta') = \alpha'^i + \beta'^i + T^i_{[ab]}\alpha'^a\beta'^b + \left(\frac{T^i_{(ab)}}{2} + \delta^i_{ab}\right)(\alpha'^a\beta'^b + \alpha'^b\beta'^a) + O((\alpha', \beta')^3).
$$

It is thus clear that if we choose

$$
\delta^i_{ab} = -\frac{T^i_{(ab)}}{2},
$$

the last piece vanishes and p'^i depends only on the antisymmetric part $T^i_{[ab]}$. From now on we will take the tensor T to be antisymmetric in a and b without loss of generality.

Let's now compute the coordinates of the inverse element

$$
g(\bar{\alpha}) \equiv g(\alpha)^{-1}.
$$

From the definition of inverse we have

$$
g(p(\bar{\alpha}, \alpha)) = g(\bar{\alpha})g(\alpha) = e = g(0).
$$

So we have the equation

$$
0 = p^i(\bar{\alpha}, \alpha) = \bar{\alpha}^i + \alpha^i + T^i_{ab}\bar{\alpha}^a\alpha^b + o((\alpha, \beta)^3).
$$

We can solve this equation iteratively for $\bar{\alpha}$ as a function of α to find

$$
\bar{\alpha}^i = -\alpha^i + T^i_{ab}\alpha^a\alpha^b + o((\alpha, \beta)^3) = -\alpha^i + o((\alpha, \beta)^3)
$$

where the last equality is because T is antisymmetric.

Let's now compute the commutator

$$
g(c(\alpha, \beta)) \equiv g^{-1}(\alpha)g^{-1}(\beta)g(\alpha)g(\beta)
$$

for α and β close to the origin. Let's rewrite the product of the first two group elements as such

$$
g^{-1}(\alpha)g^{-1}(\beta) = (g(\beta)g(\alpha))^{-1} = g(\bar{p}(\beta, \alpha)).
$$

In this way the commutator is given by

$$
c^i(\alpha, \beta) = p^i(\bar{p}(\beta, \alpha), p(\alpha, \beta))).
$$

Now we just have to expand this product using the formulas we found previously (for sake of notation, we will omit the $o((\alpha, \beta)^3)$ in every equality)

$$
\begin{aligned}
c^i(\alpha, \beta) &= p^i(\bar{p}(\beta, \alpha), p(\alpha, \beta)) \\
&= \bar{p}^i(\beta, \alpha) + p^i(\alpha, \beta) + T^i_{ab}\bar{p}^a(\beta, \alpha)p^b(\alpha, \beta) \\
&= -p^i(\beta, \alpha) + p(\alpha, \beta) + T^i_{ab}(-p^a(\beta, \alpha))p^b(\alpha, \beta) \\
&= -(\beta^i + \alpha^i + T^i_{ab}\beta^a\alpha^b) + (\alpha^i + \beta^i + T^i_{ab}\alpha^a\beta^b) + T^i_{ab}(-(\beta^a + \alpha^a))(\alpha^b + \beta^b) \\
&= (T^i_{ab} - T^i_{ba})\beta^a\alpha^b
\end{aligned}
$$

where in the last line we have again dropped some terms due to the antisymmetry of T. The computation we just made proves that $c(\alpha, \beta)$ close to the identity is linear in its arguments and antisymmetric.

The last thing we need to show to identify c with the Lie product is that it satisfies the Jacobi identity. We will now see that this comes from the associative property of the group

$$p^i(\alpha, p(\beta, \gamma)) = p^i(p(\alpha, \beta), \gamma)$$

Since the Jacobi identity involves product of two T symbols, we will need to expand p to the cubic order, that is restore the tensors B and D previously introduced. Let's expand the left-hand side of this equation

$$\begin{aligned}
p^i(\alpha, p(\beta, \gamma)) =& \alpha^i + p^i(\beta, \gamma) + T^i_{ab}\alpha^a p^b(\beta, \gamma) + B^i_{abc}\alpha^a\alpha^b p^c(\beta, \gamma) + D^i_{abc}\alpha^a p^b(\beta, \gamma)p^c(\beta, \gamma) \\
=& \alpha^i + (\beta^i + \gamma^i + T^i_{ab}\beta^a\gamma^b + B^i_{abc}\beta^a\beta^b\gamma^c + D^i_{abc}\beta^a\gamma^b\gamma^c) \\
&+ T^i_{ab}\alpha^a(\beta^b + \gamma^b + T^b_{cd}\beta^c\gamma^d) \\
&+ B^i_{abc}\alpha^a\alpha^b(\beta^c + \gamma^c) + D^i_{abc}\alpha^a(\beta^b + \gamma^b)(\beta^c + \gamma^c) \\
=& \alpha^i + \beta^i + \gamma^i + T^i_{ab}\beta^a\gamma^b + T^i_{ab}\alpha^a\beta^b + T^i_{ab}\alpha^a\gamma^b + T^i_{ab}T^b_{cd}\alpha^a\beta^c\gamma^d \\
&+ B^i_{abc}\beta^a\beta^b\gamma^c + B^i_{abc}\alpha^a\alpha^b\beta^c + B^i_{abc}\alpha^a\alpha^b\gamma^c + D^i_{abc}\beta^a\gamma^b\gamma^c + D^i_{abc}\alpha^a\beta^b\beta^c + D^i_{abc}\alpha^a\gamma^b\gamma^c \\
&+ 2D^i_{abc}\alpha^a\beta^b\gamma^c.
\end{aligned}$$

In the last step we have used explicitly the symmetry properties of B and D. The right hand side gives a very similar result

$$\begin{aligned}
p^i(p(\alpha, \beta), \gamma) =& \alpha^i + \beta^i + \gamma^i + T^i_{ab}\beta^a\gamma^b + T^i_{ab}\alpha^a\beta^b + T^i_{ab}\alpha^a\gamma^b + T^i_{ab}T^a_{cd}\alpha^b\beta^c\gamma^d \\
&+ B^i_{abc}\beta^a\beta^b\gamma^c + B^i_{abc}\alpha^a\alpha^b\beta^c + B^i_{abc}\alpha^a\alpha^b\gamma^c + D^i_{abc}\beta^a\gamma^b\gamma^c + D^i_{abc}\alpha^a\beta^b\beta^c + D^i_{abc}\alpha^a\gamma^b\gamma^c \\
&+ 2B^i_{abc}\alpha^a\beta^b\gamma^c.
\end{aligned}$$

Many of the terms simplify leading to the equation

$$T^i_{ab}T^b_{cd}\alpha^a\beta^c\gamma^d - T^i_{ab}T^a_{cd}\alpha^b\beta^c\gamma^d = 2B^i_{abc}\alpha^a\beta^b\gamma^c - 2D^i_{abc}\alpha^a\beta^b\gamma^c.$$

Equivalently, collecting α, β and γ

$$T^i_{ak}T^k_{bc} - T^i_{kc}T^k_{ab} = 2B^i_{abc} - 2D^i_{abc}.$$

The trick to recover the Jacobi identity is now to do an antisymmetric sum over all the permutation of the indices a, b and c, that is, calling this equation $eq(a, b, c)$, to compute

$$eq(a, b, c) + eq(b, c, a) + eq(c, a, b) - eq(b, a, c) - eq(a, c, b) - eq(c, b, a).$$

In this way, the terms on the right hand side will sum out to zero, since they are symmetric on the indices (a, b) and (b, c) respectively. The left hand also simplifies greatly by using the antisymmetry property of T, leaving only three terms

$$T^i_{ka}T^k_{bc} + T^i_{kb}T^k_{ca} + T^i_{kc}T^k_{ab} = 0$$

that is, the Jacobi identity.

Exercise 2

We now show how one can build an irreducible representation of the Algebra of $SU(2)$ and therefore also a representation of the Group. Given the commutation relations

$$\left[T^a, T^b\right] = i\epsilon_{abc}T^c,$$

one can compute the following

$$\begin{aligned}
\left[T^\pm, T^\pm\right] &= \frac{1}{2}\left[T^1 \pm iT^2, T^1 \pm iT^2\right] = \pm\frac{i}{2}\left[T^1, T^2\right] \pm \frac{i}{2}\left[T^2, T^1\right] = 0, \\
\left[T^+, T^-\right] &= \frac{1}{2}\left[T^1 + iT^2, T^1 - iT^2\right] = -\frac{i}{2}\left[T^1, T^2\right] + \frac{i}{2}\left[T^2, T^1\right] = T^3, \\
\left[T^3, T^\pm\right] &= \frac{1}{\sqrt{2}}\left[T^3, T^1 \pm iT^2\right] = \frac{1}{\sqrt{2}}\left[T^3, T^1\right] \pm \frac{i}{\sqrt{2}}\left[T^3, T^2\right] = \frac{iT^2 \pm T^1}{\sqrt{2}} = \pm T^\pm.
\end{aligned}$$

It's easy to show that the sum of squared generators commutes with all the generators

$$\left[\sum_{a=1}^{3} T^a T^a, T^b\right] = \sum_{a=1}^{3}\left(T^a\left[T^a, T^b\right] + \left[T^a, T^b\right] T^a\right) = i\epsilon_{abc}T^aT^c + i\epsilon_{abc}T^cT^a$$
$$= i\epsilon_{abc}T^aT^c - i\epsilon_{cba}T^cT^a = 0.$$

The operator $J^2 = \sum_{a=1}^{3} T^a T^a$ commutes with all the generators of the Algebra, therefore commutes with the whole Group. In an irreducible representation Ψ one can use the Schur's Lemma to prove that J^2 has to be a multiple of the identity:

$$\Psi : T^a \longrightarrow \tau^a \qquad \Psi : J^2 \longrightarrow \sum_{a=1}^{3} \tau^a\tau^a = \mu^2 \times 1,$$

where μ is some constant that we will determine in the following.
Let us consider an irreducible representation where generators are represented by $\tau^\pm$, τ^3, $\tau^a\tau^a = \mu^2 \times 1$, and let us consider inside the vector space an eigenvector $|m\rangle$ of the generator τ^3 relative to the eigenvalue m:

$$\tau^3|m\rangle = m\,|m\rangle.$$

The action of one of the other generators $\tau^\pm$ sends $|m\rangle$ into another vector $|m'\rangle$ which one can show to be still an eigenvector of τ^3 but with a different eigenvalue:

$$\tau^3|m'\rangle = \tau^3\tau^\pm|m\rangle = \tau^\pm\tau^3|m\rangle + [\tau^3, \tau^\pm]|m\rangle = m\tau^\pm|m\rangle \pm \tau^\pm|m\rangle = (m \pm 1)\tau^\pm|m\rangle,$$

that is to say the $\tau^\pm$ generators acting on $|m\rangle$ change its eigenvalue by one unity. This is why they are called *raising and lowering operators*. More precisely, if we call $|m \pm 1\rangle$ the state normalized to one respect to a given scalar product, then

$$|\tau^\pm|m\rangle|^2 = \langle m|(\tau^\pm)^\dagger\tau^\pm|m\rangle = \frac{1}{2}\langle m|(\tau^1)^2 + (\tau^2)^2 \pm i[\tau^1, \tau^2]|m\rangle = \frac{1}{2}\langle m|\mu^2 - (\tau^3)^2 \mp \tau^3|m\rangle = \frac{1}{2}(\mu^2 - m(m \pm 1))$$

where it has been used $(\tau^\pm)^\dagger = \tau^\mp$. Therefore the correct normalization is

$$\tau^\pm|m\rangle = \frac{1}{\sqrt{2}}\sqrt{\mu^2 - m(m \pm 1)}|m \pm 1\rangle.$$

Moreover, from the previous equalities one can argue that $\mu^2 - m(m \pm 1) \geq 0$, since we deal with a space with positive definite norm ($|\tau^\pm|m\rangle|^2 \geq 0$). At the end

$$m^2 + |m| \leq \mu^2.$$

This statement has two important consequences: firstly it's a proof that μ^2 is a positive quantity, and secondly it imposes a limit on the dimension of an irreducible representation: indeed starting from a given state $|m-\rangle$ one can apply the raising operator to get another state, independent from the original one. This will increase also the value of m of one unity. If one were free to keep on applying τ^+ he would end with a violation of the inequality (note that since the Casimir operator $(\tau)^2$ is proportional to the identity, its eigenvalue μ^2 is constant, i.e. does not depend on m). Hence the action of the raising operator has to give a null state at a certain point. This happens only when $m(m+1) = m_{max}(m_{max}+1) = \mu^2$. Starting from the state $|m_{max}\rangle$ one can apply the lowering operator to decrease the value of m. As before after a finite number of steps one has to find a null state

$$(\tau^-)^{n+1}|m_{max}\rangle \propto \tau^-|m_{max} - n\rangle = 0 \qquad \text{for some } n,$$

and this will happen when $(m-n)(m-n-1) = m_{min}(m_{min}-1) = \mu^2$. Matching the two relations one finds

$$m_{min}(m_{min} - 1) = m_{max}(m_{max} + 1) \implies m_{max} = -m_{min}.$$

Moreover m_{min} has been obtained starting from m_{max} with an integer number of steps equal to $2m_{max}+1$. This restricts the value of m_{max} to be a positive integer or semi-integer. Summarizing, using the notation $m_{max} = j$, an irreducible representation of the Algebra of $SU(2)$ is characterized by

- A vector space with dimension $2j+1$ with a basis given by the eigenvectors of τ^3:

$$\{|m\rangle\}, \qquad -j \leq m \leq j.$$

- The generators on this vector space are represented as follows

$$\begin{aligned}
&\tau^3|m\rangle = m|m\rangle, \\
&\sum_{a=1}^{3} \tau^a \tau^a |m\rangle = \mu^2|m\rangle = j(j+1)|m\rangle, \\
&\tau^{\pm}|m\rangle = \frac{1}{\sqrt{2}}\sqrt{j(j+1) - m(m\pm 1)}|m \pm 1\rangle.
\end{aligned}$$

As already said, these are representation of the algebra and therefore also of the $SU(2)$ group. Not all of them are representations of $SO(3)$. The problem arises when one tries to pass from the algebra (which is somehow a local representation of the group) to a global representation of the group. $SO(3)$ has indeed the property that a rotation of 2π around any axis must coincide with the identity. This restricts the value of j to be only integer (we will see it explicitly in some example).

Finally one can consider some representation:

- $j = 0$ is the trivial representation and is called *scalar representation.*
- $j = 1/2$ is the first non trivial one. It's only a representation of $SU(2)$ and is called *spinorial representation.* It's composed by two states labelled by the value of j and m: $|j = 1/2, m = \pm 1/2\rangle$.
- $j = 1$ is a representation of both groups. It is called *vectorial representation* and corresponds to the adjoint of $SU(2)$ or the fundamental of $SO(3)$. A basis for this representation is given by three states labelled by

$$|1, 1\rangle,\ |1, 0\rangle,\ |1, -1\rangle.$$

Exercise 3

- By definition of direct sum we can write D in block diagonal form,

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

It then follows trivially that D is a representation (in matrix products of D's, D_1's and D_2's will never mix up, and since individually D_1 and D_2 are representations, so will D be).

It is also clear that the vector $v_1 \oplus v_2 = (v_1, v_2)$ has $\dim V_1 + \dim V_2$ components. So

$$\dim V_1 \oplus V_2 = \dim V_1 + \dim V_2.$$

For the final part of the question note that we can write A in blocks according to the V_1 and V_2 subspaces,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{1}$$

By hypothesis the two matrices

$$AD = \begin{pmatrix} A_{11}D_1 & A_{12}D_2 \\ A_{21}D_1 & A_{22}D_2 \end{pmatrix}, \qquad DA = \begin{pmatrix} D_1A_{11} & D_1A_{12} \\ D_2A_{21} & D_2A_{22} \end{pmatrix}. \tag{2}$$

are equal. Given that D_1 and D_2 are inequivalent, the equality of the off-diagonal elements $A_{12}D_2 = D_1A_{12}$ and $A_{21}D_1 = D_2A_{21}$ imply, by the second Shur's lemma, $A_{12} = A_{21} = 0$. Given that D_1 and D_2 are irreducible, the equality of the diagonal elements $A_{11}D_1 = D_1A_{11}$, $A_{22}D_2 = D_2A_{22}$ imply, by the first Shur's lemma, that $A_{11} = \lambda_1 I$ and $A_{22} = \lambda_2 I$.

- Given two vector spaces V_1, V_2, with vectors $|v_1\rangle$, $|v_2\rangle$, the tensor product of the two is the set of all possible pairs:

$$V_1 \otimes V_2 = \{|v_1\rangle \otimes |v_2\rangle\,,\ \text{where } v_i \in V_i\}\,.$$

Moreover, the tensor product is distributive,

$$(|v_1\rangle + |w_1\rangle) \otimes |v_2\rangle = |v_1\rangle \otimes |v_2\rangle + |w_1\rangle \otimes |v_2\rangle, \qquad |v_1\rangle \otimes (|v_2\rangle + \mid w_2\rangle) = |v_1\rangle \otimes |v_2\rangle + |v_1\rangle \otimes |w_2\rangle.$$

In addition it can be shown that a basis of the tensor product of two vector spaces is given by all the possible pairs obtained by taking one element from the basis of the first vector space and one element from the basis of the second vector space.

The representation acting on the tensor product space is called tensor product representation, and it is easy to show that indeed it is a true representation of the group (even if, in general, it is reducible). Denoting by $D^1(g)$ and $D^2(g)$ two representations of the same element g of a given group $\mathcal{G}$, acting on vector spaces V_1 and V_2, the tensor product representation $D^1(g) \otimes D^2(g) \equiv D^{1\otimes 2}(g)$, has the following properties:

$$\begin{aligned} D^{1\otimes 2}(g_a)D^{1\otimes 2}(g_b) &\equiv (D^1(g_a) \otimes D^2(g_a))(D^1(g_b) \otimes D^2(g_b)) = D^1(g_a)D^1(g_b) \otimes D^2(g_a)D^2(g_b) \\ &= D^1(g_a \circ g_b) \otimes D^2(g_a \circ g_b) = D^{1\otimes 2}(g_a \circ g_b), \\ D^{1\otimes 2}(e) &= D^1(e) \otimes D^2(e) = 1_{V_1} \otimes 1_{V_2} = 1_V\,, \end{aligned}$$

where V is the tensor product space $V = V_1 \otimes V_2$.
In passing from first to second line it has been employed the fact that D^1 and D^2 act on *different* vector spaces, thus they commute (note that this is true even if D^1 and D^2 are two copies of *the same* representation). The system above shows that the tensor product representation is a representation of $\mathcal{G}$.
It is also possible to build explicitly the generators of the group in the tensor product representation. Denoting as $(t_1^a)_{ij}$ and $(t_2^a)_{xy}$ the generators in representations D^1 and D^2 respectively, one can write down the expression for an element near to the identity in representation $D^{1\otimes 2}$ as

$$\begin{aligned} \left[D^1(\alpha)\right]_{ij} \left[D^2(\alpha)\right]_{xy} &= \left[D^{1\otimes 2}(\alpha)\right]_{ijxy} = [\delta_{ij} + i\alpha^a (t_1^a)_{ij}][\delta_{xy} + i\alpha^a (t_2^a)_{xy}] + O(a^2) \\ &= \delta_{ij}\delta_{xy} + i\alpha^a \left[(t_1^a)_{ij}\delta_{xy} + \delta_{ij}(t_2^a)_{xy}\right] + O(\alpha^2), \end{aligned}$$

which can be written in tensor product notation as

$$\begin{aligned} D^1(\alpha) \otimes D^2(\alpha) &= D^{1\otimes 2}(\alpha) = [1_{V_1} + i\alpha^a t_1^a] \otimes [1_{V_2} + i\alpha^a t_2^a] + O(a^2) \\ &= 1_V + i\alpha^a \left[t_1^a \otimes 1_{V_2} + 1_{V_1} \otimes t_2^a\right] + O(\alpha^2). \end{aligned}$$

The operators in squared parentheses are the generators in the tensor product representation.